Uncertainty Propagation

Very often you will need to calculate a quantity R (the result) from a set of measurements of N other quantities: $X_1, X_2, \ldots, X_N$ -- in other words, $R = f(X_1, X_2, \ldots, X_N)$. The question then is how to determine the uncertainty δR from your uncertainties of the N measured quantities. The uncertainty δR is obtained by combining the individual uncertainties δX_i , whether arising from random or systematic uncertainty, according to the following general formula:

$$\delta R = \sqrt{\sum_{i=1}^{N} \left(\delta X_i \frac{\partial R}{\partial X_i} \right)^2}$$

where $\partial R / \partial X_i$ denotes a partial derivative. Depending on the function $R = f(X_1, X_2, \ldots, X_N)$, this can be a complicated process. Thus, some general results are provided in the table below for the case where R is calculated from only two measurements: $X \pm \delta X$ and $Y \pm \delta Y$. (The formulas can be generalized to any number of measured quantities.)

The formula for δR depends on the form of the equation used to calculate R, where $R = f(X, Y)$. Below, a, b, and c are constants (i.e., numbers with no uncertainty).

Form of the equation from which R is being calculated	Formula for calculating the uncertainty δR
(1) Sum: $R = X + Y$	$\delta R = \sqrt{(\delta X)^2 + (\delta Y)^2}$
(2) Generalized sum: $R = aX + bY + c$	$\delta R = \sqrt{(a \cdot \delta X)^2 + (b \cdot \delta Y)^2}$
(3) Product: $R = XY$	$\frac{\delta R}{R} = \sqrt{\left(\frac{\delta X}{X}\right)^2 + \left(\frac{\delta Y}{Y}\right)^2}$ or $\delta R = R\sqrt{\left(\frac{\delta X}{X}\right)^2 + \left(\frac{\delta Y}{Y}\right)^2}$
(4) Product of powers: $R = cX^a Y^b$	$\frac{\delta R}{R} = \sqrt{\left(a \cdot \frac{\delta X}{X}\right)^2 + \left(b \cdot \frac{\delta Y}{Y}\right)^2}$
(5) Any general function $R = f(X)$	$\delta R = \left\lvert \delta X \cdot \frac{dR}{dX} \right\rvert$

Note that, when dealing with sums, one calculates the **absolute uncertainty**, δR , directly; when dealing with products, the **relative uncertainty**, $\delta R / R$, becomes more convenient.